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On the essential maximal numerical range

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1. Introduction

In [4], STAMPFLI introduced the concept of maximal numerical range and used it to derive an identity for the norm of a derivation on $\mathscr{B}(\mathfrak{H})$. If T is a bounded operator on a Hilbert space $\mathfrak{H}$, then the *maximal numerical range* of T, denoted by $W_0(T)$, is defined to be the set

$$\{\lambda: (Tx_n, x_n) \to \lambda \text{ where } \|x_n\| = 1 \text{ and } \|Tx_n\| \to \|T\|\}.$$

For an operator T on $\mathfrak{H}$ the inner derivation δ_T is a map on $\mathscr{B}(\mathfrak{H})$ defined by $\delta_T(X)=TX-XT$ $(X\in\mathscr{B}(\mathfrak{H}))$. Stampfli showed that $\|\delta_T\|=2\inf\{\|T-\lambda\|: \lambda\in\mathbf{C}\}$, and $\|\delta_T\|=2\|T\|$ if and only if $0\in W_0(T)$.

In the present paper, we consider the analogous concept called *essential maximal numerical range* to derive the norm of an inner derivation on the Calkin algebra. Let $T\in\mathscr{B}(\mathfrak{H})$ and t be the image of T in the Calkin algebra $\mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$. The inner derivation d_t on $\mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$ is defined by $d_t(x)=tx-xt$. The essential maximal numerical range of T, denoted by $\text{ess } W_0(T)$, is defined to be the set

$$\{\lambda: (Tx_n, x_n) \to \lambda \text{ where } \|x_n\| = 1,\ x_n \to 0 \text{ weakly and } \|Tx_n\| \to \|t\|\}.$$

We shall see that $\|d_t\|=2\inf\{\|t-\lambda\|: \lambda\in\mathbf{C}\}$ and $\|d_t\|=2\|t\|$ if and only if $0\in\text{ess } W_0(T)$. Also, we shall show that $W_0(T)=\text{ess } W_0(T)$ under the following mild condition: $\|Tx\|\neq\|T\|$ for every unit vector x. In the final section we consider the maximal numerical range $V_0(T)$ for an element T in a general C^*-algebra and we show that $V_0(T)=W_0(T)$ if $T\in\mathscr{B}(\mathfrak{H})$ and $V_0(t)=\text{ess } W_0(T)$ where t is the image of $T\in\mathscr{B}(\mathfrak{H})$ in the Calkin algebra.

To close this introduction, we state and prove two technical but simple lemmas which will be used several times in the following sections. Recall that the essential

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norm of $T \in \mathscr{B}(\mathfrak{H})$, denoted by $\|T\|_e$, is $\inf\{\|T+K\|: K$ is compact$\}$. Note that $\|T\|_e = \|t\|$ where t is the image of T in the Calkin algebra.

Lemma 1.1. *If* $\|x_n\|=1$ *and* $x_n \to 0$ *weakly, then* $\limsup \|Tx_n\| \leqq \|T\|_e$.

Proof. For every compact operator K, $\|Tx_n\| \leqq \|T+K\| + \|Kx_n\|$. Since $\|Kx_n\| \to 0$, we have $\limsup \|Tx_n\| \leqq \|T+K\|$. Therefore the lemma follows.

Lemma 1.2. *If* $T \in \mathscr{B}(\mathfrak{H})$, *then there exists an orthonormal sequence* $\{x_n\}$ *such that* $\|Tx_n\| \to \|T\|_e$. *Furthermore, if P is an infinite rank prejection and* $TP=T$, *then we can choose* $\{x_n\}$ *so that the additional condition* $Px_n = x_n$ *for all n is satisfied.*

Proof. Suppose $x_1, x_2, \ldots, x_{k-1}$ have been constructed so that $Px_n = x_n$ and $\|Tx_n\| \geqq \|T\|_e - n^{-1}$ for $n=1, \ldots, k-1$. Let E be the projection onto the linear span of $x_1, \ldots, x_{k-1}$. Then $\|T(I-E)P\| = \|T(I-E)\| \geqq \|T(I-E)\|_e = \|T\|_e$. Hence there exists a unit vector x_k such that $(I-E)Px_k = x_k$ and $\|Tx_k\| \geqq \|T\|_e - k^{-1}$. The sequence $\{x_n\}$ constructed as above is the required one.

2. Essential maximal numerical ranges

The following proposition is similar to Theorem 5.1 in [2].

Proposition 2.1. *Let* $T \in \mathscr{B}(\mathfrak{H})$ *and* $\lambda \in \mathbf{C}$. *Then the following conditions are equivalent:*

(1) *There exists an orthonormal sequence* $\{x_n\}$ *in* $\mathfrak{H}$ *such that* $\|Tx_n\| \to \|T\|_e$ *and* $(Tx_n, x_n) \to \lambda$.

(2) *There exists a sequence* $\{x_n\}$ *of unit vectors such that* $x_n \to 0$ *weakly,* $\|Tx_n\| \to \|T\|_e$ *and* $(Tx_n, x_n) \to \lambda$.

(3) *There is a projection P of infinite rank such that* $PTP - \lambda P$ *is compact and* $\|TP\|_e = \|T\|_e$.

Proof. That (1) implies (2) is obvious.

(2)$\Rightarrow$(1): Suppose that $\{y_n\}$ is a sequence of unit vectors such that $y_n \to 0$ weakly, $\|Ty_n\| \to \|T\|_e$ and $(Ty_n, y_n) \to \lambda$. We construct an orthonormal sequence $\{x_n\}$ such that $\|Tx_n\| \geqq \|T\|_e - n^{-1}$ and $|(Tx_n, x_n) - \lambda| < n^{-1}$ as follows. Assume that $x_1, \ldots, x_{k-1}$ have been constructed. Let E be the projection onto the subspace spanned by $x_1, \ldots, x_{k-1}$. Then $\|Ey_n\| \to 0$ as $n \to \infty$. Let $z_n = \|(I-E)y_n\|^{-1}(I-E)y_n$. (Note that $(I-E)y_n \neq 0$ and hence z_n is well defined when n is large enough.) We have $\|z_n - y_n\| \to 0$. Hence $\|Tz_n\| \geqq \|T\|_e - k^{-1}$ and $|(Tz_n, z_n) - \lambda| < k^{-1}$ for some large n. Let x_k be such a z_n.

(1)$\Rightarrow$(3): Assume that (1) holds. By the proof of Theorem 5.1 in [2], we can choose a subsequence $\{y_n\}$ of $\{x_n\}$ such that

$$\sum_{m,n} |((T-\lambda)y_n, y_m)|^2 < \infty .$$

Let P be the projection onto the subspace spanned by $\{y_n\}$. Then $PTP-\lambda P$ is a Hilbert—Schmidt operator and hence compact. Since $\{y_n\}$ is orthonormal, by Lemma 1.1, $\|TP\|_e \geqq \limsup \|(TP)y_n\|$. Hence $\|TP\|_e = \|T\|_e$.

(3)$\Rightarrow$(1): Assume that (3) holds. By Lemma 1.2, there exists an orthonormal sequence $\{x_n\}$ such that $Px_n = x_n$ for all n and $\|Tx_n\| \to \|TP\|_e = \|T\|_e$. Since $PTP = \lambda P + K$ where K is compact, we have $(Tx_n, x_n) = \lambda + (Kx_n, x_n) \to \lambda$ as $n \to \infty$. (Note that, since $x_n \to 0$ weakly and K is compact, we have $\|Kx_n\| \to 0$.) Hence (1) holds.

The proof is complete.

Definition. Let $T \in \mathscr{B}(\mathfrak{H})$. The *essential maximal numerical range* of T, denoted by ess $W_0(T)$, is defined to be the set of all those $\lambda \in \mathbf{C}$ satisfying one of the conditions in Proposition 2.1.

Remark. By Lemma 1.2, we see that ess $W_0(T)$ is always non-empty. Obviously, ess $W_0(T) =$ ess $W_0(T+K)$ if K is a compact operator.

By condition (2) in Proposition 2.1, we can follow the argument of Lemma 2 in [4] to prove the convexity of ess $W_0(T)$. Thus we obtain:

Proposition 2.2. *The set* ess $W_0(T)$ *is non empty, compact, convex and contained in the essential numerical range of* T.

The following proposition is simple but useful.

Proposition 2.3. *Suppose that* $T \in \mathscr{B}(\mathfrak{H})$ *and* U *is a neighborhood of* ess $W_0(T)$. *Then there exists* $\delta > 0$ *and a subspace* $\mathfrak{M}$ *of* $\mathfrak{H}$ *of finite codimension such that* $x \in \mathfrak{M}$, $\|x\| = 1$ *and* $\|Tx\| \geqq \|T\|_e - \delta$ *imply* $(Tx, x) \in U$.

Proof. We may assume that U is open. Suppose that no such $\mathfrak{M}$ and δ exist. Then we can construct an orthonormal sequence $\{x_n\}$ such that $\|Tx\|_n \to \|T\|_e$ and $(Tx_n, x_n) \notin U$. Let $\{y_n\}$ be a subsequence of $\{x_n\}$ such that the limit $\lambda = \lim_{n\to\infty} (Ty_n, y_n)$ exists. Then $\lambda \notin U$. This is impossible because by definition we have $\lambda \in$ ess $W_0(T)$.

A consequence of the above proposition is the upper semicontinuity of the map $T \mapsto$ ess $W_0(T)$. (This result resembles Theorem 6 in [4].)

Corollary 2.4. *Let* $A \in \mathscr{B}(\mathfrak{H})$ *and let* U *be a neighborhood of* ess $W_0(A)$. *Then there exists* $\delta > 0$ *such that* $T \in \mathscr{B}(\mathfrak{H})$ *and* $\|T-A\|_e < \delta$ *imply* ess $W_0(T) \subseteqq U$.

Proof. We may choose a neighborhood V of ess $W_0(T)$ such that $V + \{\lambda \in \mathbf{C}: |\lambda| \leqq \varepsilon\} \subseteqq U$ for some positive number ε. By Proposition 2.3, there is a subspace $\mathfrak{M}$ of finite codimension and $\delta>0$ such that $x\in\mathfrak{M}$, $\|x\|=1$ and $\|Ax\|\geqq\geqq\|A\|_e-4\delta$ imply $(Ax, x)\in V$. We may assume that $2\delta<\varepsilon$. Suppose that $\|T-A\|_e<\delta$ and $\lambda\in\operatorname{ess} W_0(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $x_n\to 0$ weakly, $\|Tx_n\|\to\|T\|_e$ and $(Tx_n, x_n)\to\lambda$. When n is sufficiently large, we have $\|Tx_n\|>\|T\|_e-\delta$, $\|(T-A)x_n\|\leqq\|T-A\|_e+\delta\leqq 2\delta$ (by Lemma 1.1) and hence $\|Ax_n\|\geqq\geqq\|Tx_n\|-\|(T-A)x_n\|>\|T\|_e-3\delta>\|A\|_e-4\delta$. Let P be the projection onto $\mathfrak{M}$. Since $x_n\to 0$ weakly and $I-P$ is a finite rank projection, we have $\|Px_n-x_n\|\to 0$. Let $y_n=\|Px_n\|^{-1}Px_n$. Then $y_n\in\mathfrak{M}$ and $\|y_n-x_n\|\to 0$. Therefore $\|Ay_n\|\geqq\|A\|_e-4\delta$ and hence $(Ay_n, y_n)\in V$ when n is sufficiently large. By Lemma 1.1, when n is large enough, $\|(T-A)y_n\|\leqq\|T-A\|_e+\delta\leqq 2\delta$ and hence $(Ty_n, y_n)\in U$. With no loss of generality, we may assume that U is closed at the very beginning. Therefore $\lambda=\lim\,(Tx_n, x_n)=\lim\,(Ty_n, y_n)\in U$. The proof is complete.

3. The norm of an inner derivation on the Calkin algebra

Let $T\in\mathscr{B}(\mathfrak{H})$ and t be the image of T in the Calkin algebra $\mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$. Recall that d_t is the derivation defined on $\mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$ given by $d_t(x)=tx-xt$. The main result of the present section is the following identity:

$$\|d_t\| = 2\inf\{\|T-\lambda\|_e\colon\ \lambda\in\mathbf{C}\}.$$

Proposition 3.1. *If $\lambda\in\operatorname{ess} W_0(T)$, then $\|d_t\|\geqq 2(\|T\|_e-|\lambda|)$.*

Proof. By Proposition 2.1, there exists a projection P of infinite rank such that $ptp=\lambda p$ and $\|tp\|=\|t\|$. (Again, p is the image of P in the Calkin algebra.) Hence

$$\|d_t\| \geqq \|d_t(2p-1)\| = \|t(2p-1)-(2p-1)t\| =$$
$$= 2\|tp-pt\| \geqq 2\|tp-ptp\| \geqq 2(\|tp\|-\|ptp\|) = 2(\|t\|-|\lambda|).$$

The proof is complete.

Proposition 3.2. *We have $0\in\operatorname{ess} W_0(T)$ if and only if $\|T\|_e\leqq\|T-\lambda\|_e$ for all $\lambda\in\mathbf{C}$.*

Proof. If $0\in\operatorname{ess} W_0(T)$, then, by Proposition 3.1, we have $2\|T\|_e\leqq\|d_t\|\leqq\leqq 2\|T-\lambda\|_e$ for all $\lambda\in\mathbf{C}$. Conversely, suppose that $0\notin\operatorname{ess} W_0(T)$. Then, by a suitable scalar multiple of T, we may assume that $\operatorname{Re}\lambda\geqq 2\varepsilon$ $(\lambda\in\operatorname{ess} W_0(T))$ for some $\varepsilon>0$. By Proposition 2.3, there exist $\delta>0$ and a subspace $\mathfrak{M}$ of finite codimension such that $x\in\mathfrak{M}$, $\|x\|=1$ and $\|Tx\|\geqq\|T\|_e-3\delta$ imply $\operatorname{Re}(Tx, x)\geqq\varepsilon$. We may assume that $\delta\leqq\varepsilon$. Let $\{x_n\}$ be an orthonormal sequence in $\mathfrak{M}$ such that $\|(T-\delta)x_n\|\to\to\|T-\delta\|_e$. (The existence of such a sequence follows from Lemma 1.2.) For sufficiently large n, we have $\|(T-\delta)x_n\| \geqq \|T-\delta\|_e - \delta$ and hence $\|Tx_n\| \geqq \|T\|_e - 3\delta$. Therefore, when n is large enough, we have $\mathrm{Re}\,(Tx_n, x_n) \geqq \delta$ and hence

$$\|(T-\delta)x_n\|^2 = \|Tx_n\|^2 - 2\delta\,\mathrm{Re}\,(Tx_n, x_n) + \delta^2 \leqq \|Tx_n\|^2 - 2\delta^2 + \delta^2 = \|Tx_n\|^2 - \delta^2.$$

Let $n \to \infty$. Then we get $\|T-\delta\|_e^2 \leqq \|T\|_e^2 - \delta^2$. Thus $\|T-\delta\|_e < \|T\|_e$. Therefore, if $\|T\|_e \leqq \|T-\lambda\|_e$ for all $\lambda \in \mathbf{C}$, then we have $0 \in \mathrm{ess}\, W_0(T)$.

Theorem 3.3. *Suppose that $T \in \mathscr{B}(\mathfrak{H})$ and t is the image of T in the Calkin algebra. Then* $\|d_t\| = 2 \inf \{\|T-\lambda\|_e \colon \lambda \in \mathbf{C}\}$.

Proof. It is easy to see that there exists some $\lambda_0 \in \mathbf{C}$ such that

$$\|T-\lambda_0\|_e = \inf \{\|T-\lambda\|_e \colon \lambda \in \mathbf{C}\}.$$

By Proposition 3.2, we have $0 \in \mathrm{ess}\, W_0(T-\lambda_0)$. Hence, by Proposition 3.1, $\|d_t\| = \|d_{t-\lambda_0}\| \geqq 2\|T-\lambda_0\|_e$. Therefore the theorem is valid.

Corollary 3.4. $\|d_t\| = 2\|t\|$ *if and only if* $0 \in \mathrm{ess}\, W_0(T)$.

4. Relation between $W_0(T)$ and ess $W_0(T)$.

Let $T \in \mathscr{B}(\mathfrak{H})$. Then the following proposition follows from the definitions of $W_0(T)$ and ess $W_0(T)$.

Proposition 4.1. *If* $\|T\| = \|T\|_e$, *then* $\mathrm{ess}\, W_0(T) \subseteqq W_0(T)$.

In case $\|T\| > \|T\|_e$, nothing much can be said about the relation between $W_0(T)$ and ess $W_0(T)$. However, in that case, $W_0(T)$ is the "numerical range over the maximal vectors":

Proposition 4.2. *If* $\|T\| > \|T\|_e$, *then*

$$W_0(T) = \{(Tx, x) \colon \|x\| = 1 \text{ and } \|Tx\| = \|T\|\}.$$

Proof. Since $\|T^*T\|_e = \|T\|_e^2 < \|T^*T\|$, there is a finite rank projection P commuting with T^*T such that $\|T^*T(I-P)\| < \|T^*T\|$. Hence $\|T(I-P)\| < \|T\|$. Now the proposition follows from the following lemma.

Lemma 4.3. *If P is a projection commuting with T^*T such that $\|T(I-P)\| < \|T\|$, then $W_0(T) = W_0(TP)$.*

Proof. Note that $\|TP\| = \|T\|$. Suppose that $\lambda \in W_0(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $\|Tx_n\| \to \|T\|$ and $(Tx_n, x_n) \to \lambda$. Since $\|T\|^2 \geqq \|T^*Tx_n\| \geqq (T^*Tx_n, x_n) = \|Tx_n\|^2 \to \|T\|^2$, we have $\|T^*Tx_n\| \to \|T\|^2$. Write

$x_n=\alpha_n y_n+\beta_n z_n$ with $\|y_n\|=\|z_n\|=1$, $|\alpha_n|^2+|\beta_n|^2=1$, $Py_n=y_n$ and $Pz_n=0$. Now, since P commutes with T^*T,

$$\|T\|^2 \geqq |\alpha_n|^2\|(T^*T)^{1/2}y_n\|^2+|\beta_n|^2\|(T^*T)^{1/2}z_n\|^2 =$$
$$= \|(T^*T)^{1/2}(\alpha_n y_n)+(T^*T)^{1/2}(\beta_n z_n)\|^2 =$$
$$= \|(T^*T)^{1/2}x_n\|^2 = \|Tx_n\|^2 \to \|T\|^2.$$

Since $\|(T^*T)^{1/2}z_n\|^2=\|Tz_n\|^2=\|T(I-P)z_n\|^2\leqq\|T(I-P)\|^2<\|T\|^2$, $\lim\beta_n=0$. Hence $\|Ty_n\|\to\|T\|$ and $(Ty_n, y_n)\to\lambda$. Now it is easy to see that $\lambda\in W_0(TP)$. The proof of $W_0(TP)\subseteqq W_0(T)$ is straightforward and hence omitted.

Next we show that $W_0(T)=\text{ess } W_0(T)$ under a rather mild condition.

Proposition 4.4. *If $T\in\mathcal{B}(\mathfrak{H})$ fails to attain its norm (in the sense that $\|Tx\|\neq\|T\|\,\|x\|$ unless $x=0$), then $W_0(T)=\text{ess } W_0(T)$.*

Proof. From the proof of Proposition 4.2 and the given condition we see that $\|T\|=\|T\|_e$. Now the proposition follows from the following lemma.

Lemma 4.5. (Holmes and Kripke [3; Lemma 2]) *If $T\in\mathcal{B}(\mathfrak{H})$ fails to attain its norm and if $\{x_n\}$ is a sequence of unit vectors in $\mathfrak{H}$ such that $\|Tx_n\|\to\|T\|$, then $x_n\to 0$ weakly.*

Corollary 4.7. *If $\mathfrak{H}$ is a separable Hilbert space and $T\in\mathcal{B}(\mathfrak{H})$, then there is a compact operator K such that $W_0(T+K)=\text{ess } W_0(T)$.*

Proof. It suffices to show that there exists a compact operator K such that $T+K$ fails to attain its norm. Let the polar decomposition of T be $T=VP$ where $P=(T^*T)^{1/2}$ and V is a partial isometry. By considering eigenvalues of P, we can show that there exists a hermitian compact operator J such that $P+J$ has the following three properties: first, $P+J$ remains to be positive; second, the range of $P+J$ is in the initial space of V; third, $P+J$ has no eigenvalue greater than or equal to $\|P\|_e$. By the third property, it is easy to see that $P+J$ does not attain its norm. Let $K=VJ$. Then $T+K=V(P+J)$. If $\|(T+K)x\|=\|T+K\|\,\|x\|$, then by the second property of $P+J$, we have

$$\|(P+J)x\| = \|V(P+J)x\| = \|V(P+J)\|\,\|x\|$$
$$\geqq \|V^*V(P+J)\|\,\|x\| = \|P+J\|\,\|x\|$$

and hence $x=0$. Therefore $T+K$ does not attain its norm. The proof is complete.

5. Maximal numerical range of an element in C^*-algebra

Let $\mathscr{A}$ be a C^*-algebra with identity I and let T be an element in $\mathscr{A}$. Recall that a linear functional f on $\mathscr{A}$ is called a *state* if $f(I)=\|f\|=1$. We call a state f is *maximal* for T if $f(T^*T)=\|T\|^2$. We shall denote by $S_0(T, \mathscr{A})$ the set of all maximal states of T. It is easy to show that $S_0(T, \mathscr{A})$ is non-empty.

Definition. The *(algebraic) maximal numerical range* of an element T in a C^*-algebra $\mathscr{A}$, denoted by $V_0(T, \mathscr{A})$, is defined to be the set $\{f(T): f\in S_0(T, \mathscr{A})\}$.

Note that $V_0(T, \mathscr{A})$ is a non-empty convex compact subset of $V(T, \mathscr{A})$, the (algebraic) numerical range of T. Because of the following proposition, $V_0(T, \mathscr{A})$ can be abbreviated as $V_0(T)$.

Proposition 5.1. *If $\mathscr{A}$ is a sub-C^*-algebra of $\mathscr{B}$ containing I and T, then $V_0(T, \mathscr{A})=V_0(T, \mathscr{B})$.*

The proof of the above proposition follows from a standard argument of Hahn—Banach type and hence is omitted.

Remark. It is easy to check that $S_0(T, \mathscr{A})$ is a *face* of the state space, that is, if f and g are two states such that $\lambda f+(1-\lambda)g$ is in $S_0(T, \mathscr{A})$ for some λ with $0<\lambda<1$, then $f, g\in S_0(T, \mathscr{A})$. However, $V_0(T)$, the image of $S_0(T, \mathscr{A})$ under the evaluation map $f\to f(T)$, is in general not a face of $V(T, \mathscr{A})$. For example, if $\mathfrak{H}=\mathbf{C}^2$ and $T\in\mathscr{B}(\mathfrak{H})$ is given by the matrix $\begin{pmatrix}0&1\\0&0\end{pmatrix}$, then $V_0(T)$ is $\{0\}$ while $V(T, \mathscr{B}(\mathfrak{H}))$ is a disk centred at 0.

Proposition 5.2. *For an element T in the C^*-algebra $\mathscr{A}$, we have $V_0(T^*)=$ $=V_0(T)^*$. (For a set S in $\mathbf{C}$, we write S^* for $\{\lambda\in\mathbf{C}: \bar\lambda\in S\}$.)*

Proof. It suffices to show that $S_0(T, \mathscr{A})=S_0(T^*, \mathscr{A})$. Let f be in $S_0(T, \mathscr{A})$. By Schwarz's inequality, we have $f(T^*T)^2\leqq f((T^*T)^2)$. Hence $\|T\|^4=f(T^*T)^2\leqq$ $\leqq f((T^*T)^2)=f(T^*(TT^*)T)\leqq\|T\|^2 f(TT^*)$. Hence $f(TT^*)\geqq\|T\|^2$. Therefore $f\in S_0(T^*, \mathscr{A})$. Thus we have shown that $S_0(T^*, \mathscr{A})=S_0(T, \mathscr{A})$ and hence the proposition follows.

Now we are going to show the main result of the present section: the algebraic maximal numerical range of an operator on Hilbert space is the same as the usual one. First we need a lemma similar to Proposition 2.3.

Lemma 5.2. *If U is a neighbourhood of $W_0(T)$, then there is a positive number δ such that $(Tx, x)\in U$ for the unit vectors x satisfying $\|Tx\|\geqq\|T\|-\delta$.*

The proof is the same as that of Proposition 2.3 and hence omitted.

Remark. By using this lemma, we can show that the map $T \mapsto W_0(T)$ is upper semi-continuous.

Theorem 5.4. *If T is an operator on a Hilbert space $\mathfrak{H}$, then* $W_0(T) = V_0(T, (\mathscr{B}(\mathfrak{H}))$.

Proof. Suppose that $\lambda \in W_0(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $\|Tx_n\| \to \|T\|$ and $(Tx_n, x_n) \to \lambda$. For each n, define a state f_n on $\mathscr{B}(\mathfrak{H})$ by $f_n(A) = (Ax_n, x_n)$. By the compactness of the state space, $\{f_n\}$ has a subsequence converging in the weak*-topology to some state, say f. Then $\|T\|^2 = \lim \|Tx_n\|^2 = \lim (T^*Tx_n, x_n) = f(T^*T)$ and $\lambda = \lim (Tx_n, x_n) = \lim f_n(T) = f(T)$. Therefore $\lambda \in V_0(T, \mathscr{B}(\mathfrak{H}))$.

Conversely, suppose that $\lambda \in V_0(T)$. We assume on the contrary that $\lambda \notin W_0(T)$. Since, by Lemma 2 in [4], $W_0(T)$ is compact and convex, there is an open half-space H containing $W_0(T)$ such that $\lambda \notin H^-$, the closure of H. By Lemma 5.3, there exists a positive number δ such that $(Tx, x) \in H$ for all x with $\|x\| = 1$ and $\|Tx\|^2 \geqq \|T\|^2 - \delta$. We can choose δ small enough so that $3\delta\|T\| < \operatorname{dist}(\lambda, H)$, the distance from λ to H. It is well-known that convex combinations of vector states are dense in the state space in the weak*-topology. Hence there exists a linear functional f of the form $f(A) = \sum_n \mu_n (Ax_n, x_n)$ with $\mu_n \geqq 0$, $\sum \mu_n = 1$ and $\|x_n\| = 1$ such that $f(T^*T) \leqq \|T\|^2 - \delta^2$ and $|f(T) - \lambda| < \delta\|T\|$. Let

$$\mathscr{S} = \{n\colon \|Tx_n\|^2 \geqq \|T\|^2 - \delta\}.$$

Then

$$\|T\|^2 - \delta^2 \leqq f(T^*T) = \sum \mu_n \|Tx_n\|^2 \leqq (\|T\|^2 - \delta)\big(\sum_{n \notin \mathscr{S}} \mu_n\big) + \|T\|^2\big(\sum_{n \in \mathscr{S}} \mu_n\big) = \|T\|^2 - \delta\big(\sum_{n \notin \mathscr{S}} \mu_n\big).$$

Hence $\sum_{n \notin \mathscr{S}} \mu_n \leqq \delta$. Therefore,

$$\Big|f(T) - \sum_{n \in \mathscr{S}} \mu_n (Tx_n, x_n)\Big| = \Big|\sum_{n \notin \mathscr{S}} \mu_n (Tx_n, x_n)\Big| \leqq \delta\|T\|.$$

Let $\lambda_n = \big(\sum_{n \in \mathscr{S}} \mu_n\big)^{-1} \mu_n$. Then we have $\sum_{n \in \mathscr{S}} \lambda_n (Tx_n, x_n) \in H$ and

$$\Big|\sum_{n \in \mathscr{S}} \lambda_n (Tx_n, x_n) - \sum_{n \in \mathscr{S}} \mu_n (Tx_n, x_n)\Big| = \Big|\big(1 - \sum_{n \in \mathscr{S}} \mu_n\big)\big(\sum_{n \in \mathscr{S}} \lambda_n (Tx_n, x_n)\big)\Big| \leqq \delta\|T\|.$$

Hence $\operatorname{dist}(f(T), H) \leqq 2\delta\|T\|$. From this and $\|f(T) - \lambda| < \delta\|T\|$ we see that $\operatorname{dis}(\lambda, H) < 3\delta\|T\|$. This contradicts our choice of δ which satisfies the inequality $3\delta\|T\| < \operatorname{dist}(\lambda, H)$.

Next we prove a theorem similar to Theorem 5.4 for an element in the Calkin algebra. First we need a simple lemma.

Lemma 5.5. *Let $T \in \mathscr{B}(\mathfrak{H})$ and t be its image in the Calkin algebra. If $\|T\| = \|t\|$, then $V_0(t) \subseteqq V_0(T)$.*

Proof. Let $\lambda \in V_0(t)$. Then there is a state g on the Calkin algebra $\mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$ such that $g(t^*t)=\|t\|^2$ and $g(t)=\lambda$. Let p be the canonical projection from $\mathscr{B}(\mathfrak{H})$ to the Calkin algebra. Then $f=g\circ p$ is a state on $\mathscr{B}(\mathfrak{H})$ satisfying $f(T^*T)=\|T\|^2$ and $f(T)=\lambda$. Hence $\lambda \in V_0(T)$.

Theorem 5.6. *If $\mathfrak{H}$ is a separable Hilbert space, T is an operator on $\mathfrak{H}$ and t is its image in the Calkin algebra, then* $\operatorname{ess} W_0(T)=V_0(t, \mathscr{B}(\mathfrak{H})/\mathscr{K}(\mathfrak{H}))$.

Proof. By Corollary 4.6, there is a compact operator K such that $\|T+K\|=\|t\|$ and $\operatorname{ess} W_0(T)=W_0(T+K)$. By Theorem 5.4, $W_0(T+K)=V_0(T+K)$. By Lemma 5.5, we have $V_0(T+K)\supseteq V_0(t)$. Hence $\operatorname{ess} W_0(T)\supseteq V_0(t)$.

On the other hand, suppose $\lambda\in\operatorname{ess} W_0(T)$. Then, by Proposition 2.1, there exists a projection P in $\mathscr{B}(\mathfrak{H})$ such that its image p in the Calkin algebra satisfies $ptp=\lambda p$ and $\|tp\|=\|t\|$. Let $\mathscr{C}$ be the commutative algebra generated by $1, p$ and pt^*tp. Then it is easy to see that there is a multiplicative linear functional g on $\mathscr{C}$ such that $g(p)=g(1)=1$ and $g(pt^*tp)=\|pt^*tp\|=|tp\|^2=\|t\|^2$. Let h be a state on the Calkin algebra which extends g and let f be the functional given by $f(x)=h(pxp)$. Then it is easy to check that f is a state on the Calkin algebra, $f(t^*t)=\|t\|^2$ and $f(t)=\lambda$. Therefore $\lambda\in V_0(t)$.

Remarks 1. The above theorem can be proved in the same way as Theorem 5.4, by using Proposition 2.3 instead of Lemma 5.5. This alternative proof does not require the underlying Hilbert space $\mathfrak{H}$ to be separable. 2. Because of Theorem 5.4, many results concerning maximal numerical ranges of operators can be extended to corresponding results for maximal numerical ranges of elements in C^*-algebras. For instance, Pythagorean relation for operators in [4] becomes: if T is an element in a C^*-algebra, then there exists a unique $z_0\in\mathbf{C}$ such that $\|T-z_0\|^2+|\lambda|^2\leqq\|(T-z_0)+\lambda\|^2$ for all λ in $\mathbf{C}$; moreover, $0\in V_0(T-\lambda)$ if and only if $\lambda=z_0$.

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